



$$T(v_1) = u_1$$

$$T(v_2) = u_2$$

Def: $T: V \rightarrow V$ is called a linear transformation or a linear map if $\alpha, \beta \in F$ $x, y \in V$ (LTT)

$$\checkmark T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

$T: V \rightarrow V$ T is linear operator.

Prop. i)

$$T(0) = 0$$

$\downarrow$
 V

additive identity of V .

$T(x) = x$ T is identity transformation.

Eg: $T: P_n(\mathbb{R}) \rightarrow P_n(\mathbb{R})$ $T(p(x)) = \frac{d}{dx} p(x)$

$$x = p(x), y = q(x)$$

$$\begin{aligned} T(\alpha x + \beta y) &= T(\alpha p(x) + \beta q(x)) \\ &= \frac{d}{dx} [\alpha p(x) + \beta q(x)] \\ &= \alpha \frac{d}{dx} p(x) + \beta \frac{d}{dx} q(x) \\ &= \alpha T(p(x)) + \beta T(q(x)) \\ &= \alpha T(x) + \beta T(y) \end{aligned}$$

You can show $T: P_n(\mathbb{R}) \rightarrow \mathbb{R}$ $T(p(x)) = \int_0^1 p(x) dx$ is also a LTT.

Eg: $T: \mathbb{R}^2(\mathbb{R}) \rightarrow \mathbb{R}^2(\mathbb{R})$ $T(x, y) = (-y, x)$

$$\begin{aligned} x &= (x_1, y_1), y = (x_2, y_2) \quad T(\alpha x + \beta y) = T[\alpha(x_1, y_1) + \beta(x_2, y_2)] \\ &= T(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2) \\ &= (-\alpha y_1 - \beta y_2, \alpha x_1 + \beta x_2) \\ &= (-\alpha y_1, \alpha x_1) + (-\beta y_2, \beta x_2) \end{aligned}$$

$$\begin{aligned}
 &= (-\alpha x_1 - \beta x_2, \alpha y_1 + \beta y_2) \\
 &= (-\alpha x_1, \alpha y_1) + (-\beta x_2, \beta y_2) \\
 &= \alpha(-x_1, y_1) + \beta(-x_2, y_2) \\
 &= \alpha T(x_1, y_1) + \beta T(x_2, y_2) \\
 &= \alpha T(X) + \beta T(Y)
 \end{aligned}$$

Eg: $T: \mathbb{R}^2(\mathbb{R}) \rightarrow \mathbb{R}^2(\mathbb{R}) \quad T(x, y) = (x, 0)$

Eg: $T: \mathbb{R}^3(\mathbb{R}) \rightarrow \mathbb{R}^2(\mathbb{R}) \quad T(x, y, z) = (x+y, y+z)$

$$X = (x_1, y_1, z_1) \quad Y = (x_2, y_2, z_2)$$

$$T(\alpha X + \beta Y) = T[\alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2)]$$

$$= T(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2)$$

$$= (\alpha x_1 + \beta x_2 + \alpha y_1 + \beta y_1, \alpha y_1 + \beta y_2 + \alpha z_1 + \beta z_2)$$

$$= (\alpha(x_1 + y_1) + \beta(x_2 + y_2), \alpha(y_1 + z_1) + \beta(y_2 + z_2))$$

$$= (\alpha(x_1 + y_1), \alpha(y_1 + z_1)) + (\beta(x_2 + y_2), \beta(y_2 + z_2))$$

$$= \alpha(x_1 + y_1, y_1 + z_1) + \beta(x_2 + y_2, y_2 + z_2)$$

$$= \alpha T(x_1, y_1, z_1) + \beta T(x_2, y_2, z_2) = \alpha T(X) + \beta T(Y)$$

$$N(T) \text{ Null Space of } T = \{X \in V : TX = 0\}$$

$$R(T) \text{ Range}(T) = \{TX : X \in V\}$$

Eg: $T(x, y, z) = (x+y, y+z)$ find basis & dim of $N(T)$ & $R(T)$

$$N(T) = \{(x, y, z) : T(x, y, z) = (0, 0)\}$$

$$= \{(x, y, z) : (x+y, y+z) = (0, 0)\}$$

$$= \{(x, y, z) : x+y=0 \text{ \& } y+z=0\}$$

$$= \{(x, y, z) : x=-y \text{ \& } y=-z\}$$

$$= \{(x, -x, z) : x \in \mathbb{R}\}$$

$$\{(2, -2, 2)\}$$

$$\dim N(T) = 1$$

$$\text{Nullity} = \dim(N(T))$$

$$\text{Basis} = \{ (1, -1, 1) \}$$

$$R(T) = \{ T(x, y, z) : (x, y, z) \in \mathbb{R}^3 \}$$

$$= \{ x+y, y+z : x, y, z \in \mathbb{R} \}$$

Basis
of $\mathbb{R}^3$
T

$$(1, 0, 0)$$

$$(0, 1, 0)$$

$$(0, 0, 1)$$

$$(1, 1, 0)$$

$$(1, 1)$$

$$(0, 1)$$

$$\text{Span}(R(T)) = \{ (1, 0), (1, 1), (0, 1) \}$$

$$\text{Basis} = \{ (1, 0), (0, 1) \}$$

of $(\mathbb{R}^2)$